Topics for PhD program: Analysis and Functional Analysis

Calculus

1. **Limits**: The concept of a limit helps us understand the behavior of functions as they approach certain points, including infinity. It's foundational to calculus.
2. **Derivatives**: This topic involves studying the rate of change of a function, or how a function changes as its input changes. It's central to concepts like motion and optimization.
3. **Applications of Derivatives**: Includes understanding concepts like tangent lines, optimization problems, and related rates, where derivatives are applied to real-world situations.
4. **Integrals**: The integral is the reverse process of differentiation and is used to calculate areas under curves, volumes, and accumulated quantities.
5. **Teylor's formula**: Taylor's polynomials. The remainder in various forms: integral, Peano etc.
6. **Fundamental Theorem of Calculus**: This theorem links the concept of the derivative with the integral, stating that differentiation and integration are inverse operations.
7. **Techniques of Integration**: Various methods to compute integrals, such as substitution, integration by parts, partial fractions, and trigonometric identities.
8. **Infinite Sequences and Series**: Understanding the behavior of sequences and series, including concepts like convergence, divergence, and power series.
9. **Multivariable Calculus**: This extends calculus to functions of more than one variable, covering topics like partial derivatives, multiple integrals, and gradient fields.
10. **Vector Calculus**: Deals with vector fields, line integrals, surface integrals, and theorems like Green's Theorem, Stokes' Theorem, and the Divergence Theorem.

Reading materials:

- Thomas' Calculus Early Transcendentals, Twelfth Edition. We'll cover most of chapters 1 – 8.
- James Stewart's Calculus, 7th edition
- Frank Ayres, Elliot Mendelson, Calculus, 4th edition, Schaum's Outlines.
- Howard Anton, Calculus, John Wiley and Sons / New York – 1999.
- C. Henry Edwards, David E. Penney, Calculus (6th Edition)
- Robert Adams, Calculus A Complete Course, Pearson, Toronto, 2006
- Jon Rogawski, Calculus

Real Analysis:

1. **Sequences and Limits**: The supremum and infimum of sequences of numbers and sets, convergence of Cauchy sequences, and the properties of limit points.
2. **Uniform Continuity**: A function is uniformly continuous if small changes in the input lead to small changes in the output, uniformly across the entire domain. A detailed exploration of uniformly continuous functions, including the Lipschitz continuity.
3. **Topology of the Real Line**: Understanding open, closed, and compact sets, connectedness, and the structure of the real number line.
4. **Metric Spaces**: Generalizing distance, studying spaces equipped with a metric, and exploring concepts such as completeness and compactness in these spaces.
5. **Compactness**: Understanding the important topological concept of compact sets, which are closed and bounded in Euclidean spaces and have special properties like every sequence having a convergent subsequence.
6. **Sequences and Series of Functions**: Investigating pointwise and uniform convergence, and the interaction between limits and operations like integration and differentiation.
7. **Convergence Theorems**: The Dominated Convergence Theorem, Fatou's Lemma, and other important theorems for the interchange of limits and integrals.
8. **The Baire Category Theorem**: A foundational result in topology and analysis that describes the structure of complete metric spaces and their applications in real analysis.
9. **Lebesgue Measure and Integration (Introduction)**: The basic ideas behind Lebesgue integration, an alternative to Riemann integration that allows for a broader class of integrable functions.
10. **Implicit and Inverse Function Theorems**: The conditions under which a system of equations defines a function implicitly and the properties of such functions. Describes conditions under which a differentiable function has a differentiable inverse, and the relationship between their derivatives.

Reading materials:

- Robert G. Bartle, Introduction to Real Analysis
- Stephen Abbott, Understanding Analysis

Functional Analysis

1. **Normed Spaces**: The study of vector spaces equipped with a norm, which assigns a length or size to each vector, and the properties of these spaces such as completeness (Banach spaces).
2. **Banach Spaces**: A normed space that is complete, meaning every Cauchy sequence in the space converges to an element within the space.

3. **Inner Product Spaces**: Vector spaces equipped with an inner product, which defines angles and lengths. These spaces generalize Euclidean geometry to more abstract settings (Hilbert spaces).
4. **Hilbert Spaces**: Complete inner product spaces, central in quantum mechanics and many other fields. They include spaces like L2 spaces, which consist of square-integrable functions.
5. **Linear Operators**: Study of operators (functions between vector spaces) that preserve the linear structure, including bounded and unbounded operators, and their properties.
6. **Spectral Theory**: Analysis of the spectrum of operators, including eigenvalues and eigenvectors, and the spectral decomposition of operators, especially in Hilbert and Banach spaces.
7. **Bounded and Continuous Linear Operators**: Focus on operators that behave predictably with respect to the topology of the spaces involved, including the study of the operator norm.
8. **Dual Spaces**: The space of all continuous linear functionals on a given vector space, with important applications in optimization and the theory of distributions.
9. **Compact Operators**: Linear operators that map bounded sets to relatively compact sets. Compact operators generalize matrices and have important properties related to eigenvalues and spectral theory.
10. **Topological Properties of Operator Spaces**: The study of the topology of operator spaces, including weak, strong, and weak-* topologies, and their application in understanding convergence, continuity, and the structure of operators.

Reading materials:

- Erwin Kreyszig, Introductory functional analysis with applications
- Sheldon Axler, Measure, Integration & Real Analysis
- Marat V. Markin, Elementary Functional Analysis

Complex Analysis:

1. **Complex Numbers and Basic Operations**: The foundation of complex analysis, focusing on the properties and arithmetic of complex numbers, including addition, multiplication, division, and polar form.
2. **Analytic Functions**: Functions that are complex differentiable in some neighborhood of every point in their domain. These functions satisfy the Cauchy-Riemann equations and are central to complex analysis.
3. **Cauchy-Riemann Equations**: A set of partial differential equations that a function must satisfy in order to be analytic. They provide a condition for a function to be differentiable in the complex plane.
4. **Cauchy's Integral Theorem**: A fundamental result in complex analysis stating that the integral of an analytic function over a closed contour is zero, provided the function is analytic inside and on the contour.

5. **Cauchy's Integral Formula**: A key result that expresses the value of an analytic function inside a contour in terms of the values of the function on the contour. It is used to compute integrals and derive properties of analytic functions.
6. **Residue Theorem**: A powerful tool for evaluating contour integrals by relating the integral around a closed curve to the sum of residues of the enclosed singularities. This is widely used for computing integrals in complex analysis.
7. **Singularities and Poles**: The study of points where a complex function ceases to be analytic, such as poles (where the function blows up) and essential singularities. Understanding the types of singularities is crucial for analyzing functions.
8. **Laurent Series**: An expansion of a complex function around a singularity that includes both positive and negative powers of the variable. This is useful for analyzing functions with singularities and understanding their behavior near poles.
9. **Conformal Mapping**: The study of mappings that preserve angles and the shape of infinitesimally small structures. Conformal maps are important in fluid dynamics, aerodynamics, and other fields involving complex geometries.
10. **Riemann Surfaces**: A more advanced topic involving the geometric visualization of complex functions. Riemann surfaces allow the extension of complex functions to multi-valued functions, like the square root or logarithm, by defining them over a surface rather than a plane.

Reading materials:

- J.W. Brown, R.V. Churchill, Complex Variables and Applications, 8th Ed., McGraw-Hill.
- G. Sveshnikov, A. N. Tikhonov, The Theory Of Functions Of A Complex Variable,1982

Differential Equations

1. **First-Order Differential Equations**: These equations involve the first derivative of a function. Topics include separable equations, linear equations, exact equations, and methods like substitution to solve them.
2. **Higher-Order Differential Equations**: Equations involving derivatives of order greater than one. The study includes linear equations with constant coefficients, homogeneous and non-homogeneous equations, and methods of solving them.
3. **Linear Differential Equations**: A major class of differential equations that involve linear terms in the unknown function and its derivatives. This includes both first and higher-order linear equations, as well as the study of their solutions and superposition principle.
4. **Systems of Differential Equations**: These involve multiple interrelated differential equations. Solutions can be analyzed using methods such as eigenvalues and eigenvectors, and solutions are often found in terms of vector spaces.

5. **Laplace Transforms**: A technique used to transform a differential equation into an algebraic equation, making it easier to solve. Laplace transforms are widely used for solving initial value problems and handling discontinuities and forcing functions.
6. **Nonlinear Differential Equations**: Equations where the unknown function and its derivatives appear in nonlinear ways. These are often harder to solve, and include methods for approximation, qualitative analysis, and numerical solutions.
7. **Stability and Qualitative Analysis**: Investigating the stability of equilibrium points in dynamical systems described by differential equations. Techniques like phase portraits, Lyapunov functions, and bifurcation analysis are used.
8. **Partial Differential Equations (PDEs)**: Equations involving partial derivatives of functions of several variables. Common methods of solving PDEs include separation of variables, Fourier series, and transform methods. Applications range from physics to engineering.
9. **Boundary and Initial Value Problems**: Boundary value problems involve finding solutions to differential equations with specific values or conditions at the boundaries of the domain, while initial value problems provide conditions at a specific point in time.
10. **Numerical Methods for Differential Equations**: These methods are used to find approximate solutions to differential equations when exact solutions are difficult or impossible to obtain. Techniques include Euler’s method, Runge-Kutta methods, and finite difference methods.

Reading materials:

- Alekseĭ Fedorovich Filippov, Joel Lee Brenner, Problems in differential equations, W. H. Freeman, 1966.
- Wolfgang Walter, Ordinary Differential Equations, Springer, 1991.
- M.R. Spiegel, Laplace Transforms, Schaum’s Outlines Series, McGraw-Hill, 1965.
- Shepley L. Ross, Differential Equations, Wiley, 1984
- William F. Trench Elementary Differential Equations with Boundary Value Problems, Trinity University, Digital Commons @ Trinity Books and Monographs,

Optimization Theory

1. **Linear Programming**: Involves optimizing a linear objective function subject to linear constraints. Common methods include the Simplex method and Interior-point methods. Applications are found in resource allocation, logistics, and finance.
2. **Convex Optimization**: Focuses on problems where the objective function and constraints are convex. Convex problems are particularly important because they

have desirable properties like the existence of global optima. Key methods include gradient descent and interior-point methods.

3. **Nonlinear Optimization**: Deals with problems where the objective function or constraints are nonlinear. These problems often require more advanced techniques, such as sequential quadratic programming (SQP) or trust-region methods.
4. **Integer Programming**: Optimization problems where some or all decision variables are restricted to integer values. Applications include scheduling, resource allocation, and network design. Techniques like branch-and-bound are commonly used for solving these problems.
5. **Dynamic Programming**: A method for solving complex optimization problems by breaking them down into simpler subproblems. It is widely used in time-dependent optimization problems, such as those found in control theory, inventory management, and decision-making processes.
6. **Optimal Control Theory**: Focuses on determining a control function that will optimize the performance of a system over time. Applications are found in fields like engineering, economics, and biology, with methods including Pontryagin's Maximum Principle and the Hamilton-Jacobi-Bellman equation.
7. **Stochastic Optimization**: Deals with optimization problems involving uncertainty. The objective or constraints may depend on random variables, and techniques like Monte Carlo simulations or stochastic gradient descent are used to find solutions.
8. **Multi-objective Optimization**: Involves optimizing two or more conflicting objectives simultaneously. Solutions in this area aim to find a set of Pareto-optimal solutions, where no objective can be improved without degrading another. Methods include Pareto optimization and weighted sum approaches.

Reading materials:

- Optimization—Theory and Applications, Lamberto Cesari
- Optimization Methods, Theory and Application, Honglei Xu, Song Wang, Soon-Yi Wu

Differential Geometry and Topology

1. **Curves and Surfaces**: The study of the geometry of curves (1-dimensional objects) and surfaces (2-dimensional objects) in higher-dimensional spaces. Key concepts include arc length, curvature, and parametrization.
2. **Differential Manifolds**: A generalization of curves and surfaces to higher dimensions. These are spaces that locally resemble Euclidean space and are equipped with a smooth structure, allowing for calculus to be done on them.
3. **Tensors and Tensor Fields**: The study of tensors, which generalize vectors and matrices, and tensor fields, which describe geometric quantities like curvature, stress, and deformation on manifolds.

4. **Connections and Covariant Derivatives**: Connections provide a way of comparing vectors in different tangent spaces on a manifold. The covariant derivative measures how a vector field changes along the manifold.
5. **Riemannian Geometry**: Focuses on the study of smooth manifolds equipped with a Riemannian metric, which allows for measuring distances and angles. Key topics include geodesics (shortest paths), curvature, and the curvature tensor.
6. **Curvature**: A central concept in differential geometry, curvature describes how a curve or surface deviates from being flat. This includes Gaussian curvature (for surfaces) and sectional curvature (for higher-dimensional manifolds).
7. **Geodesics**: The "straight lines" on a curved surface or manifold, which are the shortest paths between points. Geodesics play a key role in understanding the intrinsic geometry of a manifold.
8. **The Gauss-Bonnet Theorem**: A fundamental result in differential geometry that connects the topology (the shape) of a surface to its geometry (the curvature). It relates the total curvature of a surface to its Euler characteristic.
9. **Lie Groups and Lie Algebras**: The study of smooth manifolds that also have a group structure, which is important in understanding symmetries and transformations. Lie groups are used in physics, particularly in the study of symmetry in space-time.
10. **Minimal Surfaces and Variational Problems**: The study of surfaces that locally minimize area, such as soap films. This topic connects differential geometry with variational calculus, where critical points of an energy functional represent minimal surfaces.

Reading materials:

- Andrew Pressley-Elementary Diff. Geom-2nd Ed-Springer, 2010
- Mishchenko, A. Fomenko, A Course Of Differential Geometry And Topology,1988